Closed-form Method to Evaluate Bike Braking Performance

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Abstract

The brake is an important safety device for all types of vehicles. Traditionally in calculations, aerodynamic and tire rolling resistant effects are neglected and thus the braking distance and stop time may be estimated by simplified equations. To obtain a more accurate result, the inclusion of all resistant forces is necessary. In this paper, low CG (center of gravity) vehicles such as recumbent bikes with no tip-over assumed are studied. It starts with the use of dynamic equilibrium to formulate a nonlinear dynamic equation. The coefficients representing full-brake, front-brake and rear-brake cases are then obtained. The closed-form solution for the velocity, stop time, and braking distance is then derived. Results for various cases are presented, as well as a file which can be used with programs such as Matlab in order to study the effect of the air drag coefficient, frontal area, rolling resistance, road gradient, and power consumption.

Braking and Resistant Force Equilibrium

As shown in Figure 1, the major external forces acting on a vehicle during braking are

F_a, the aerodynamic force (drag; lift not considered)
F_b, the tire braking force, front and rear (= F_{bf} + F_{br})
F_r, the tire rolling resistance, front and rear (= F_{rf} + F_{rr})
F_i, the inertia force (linear; wheel inertia not considered)
F_g, the gravitational force

The dynamic equilibrium of the system along the longitudinal (x) direction can be written as

$$F_i = F_a + F_b + F_r + F_g \tag{1}$$

Sum the moments about the front wheel contact line (point A), the normal load on the rear wheel is

$$W_r = \frac{WL_a\cos(\theta)+F_a h_a+mh\dot{v}+Wh\sin(\theta)}{L} \tag{2a}$$

Similarly, sum the moments about point B, the normal load on the front wheels is

$$W_f = \frac{WL_b\cos(\theta)-F_a h_a-mh\dot{v}-Wh\sin(\theta)}{L} \tag{2b}$$

Define μ as the peak coefficient of tire/road friction, N as the normal load on the wheel, C_s as the longitudinal stiffness of the tire during braking, and i_s as the skid. The braking force may be expressed as follows (Wong, 2008):

$$F_x = \mu N\left[1 - \frac{\mu N(1-i_s)}{4C_s i_s}\right] \quad (3)$$

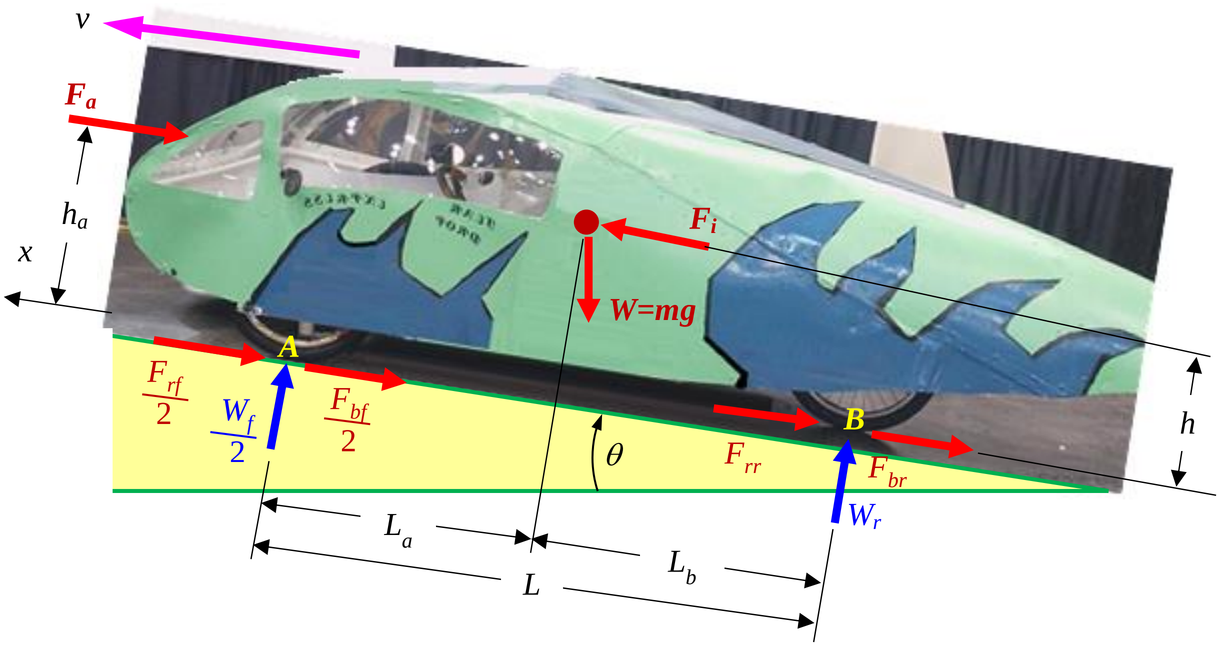

Figure 1: Force balance of a low CG tricycle during braking.

For the current study, it is assumed that the braking force is near its peak value such that the maximum braking effort may be obtained, i.e.

$$F_{bf} = \mu W_f \quad (4a)$$

$$F_{br} = \mu W_r \quad (4b)$$

Define m as the mass of bicycle plus the rider, ρ as the air density, C_D as the air drag coefficient, A_f as the frontal area, v as the forward velocity, θ as the grade angle, g as the gravitational acceleration, and f_o and f_1 as the rolling resistance coefficients (f_1 is usually very small but is included here for completeness). The air drag, rolling resistance, inertia and gravitational forces can be expressed in the following form (Gillespie, 1992; Wong, 2008):

$$F_a = \frac{\rho}{2} C_D A_f v^2 \quad (5a)$$

$$F_r = (f_o + f_1 v^2) W\cos(\theta) \quad (5b)$$

$$F_i = -m\dot{v} = -m\frac{dv}{dt} \quad (5c)$$

$$F_g = W\sin(\theta) \quad (5d)$$

Differential Equation for Braking

Summing all the forces along longitudinal (x) direction gives

$$F_i - F_{rf} - F_{rr} - F_{bf} - F_{br} - F_a - W\sin(\theta) = 0 \tag{6a}$$

or

$$F_i = F_r + F_b + F_a + W\sin(\theta) \tag{6b}$$

Substituting Eq. (5a-d) into Eq. (6b) results in

$$-m\dot{v} = (f_o + f_1 v^2)W\cos(\theta) + \mu N + \frac{\rho}{2} C_D A_f v^2 + mg\sin(\theta) \tag{7}$$

Rearranging the above equation, a nonlinear equation describing the dynamic equilibrium can be written in the following form

$$-S_1\dot{v} = S_2 + S_3 v^2 \tag{8}$$

Where S_1, S_2 and S_3 are functions of vehicle parameters, and are different for full-brake, front-brake, and rear-brake cases. To facilitate integration, Eq. (8) is re-arranged as

$$dt = -S_1 \frac{dv}{S_2 + S_3 v^2} \tag{9}$$

For the normal condition the **full-brake case** (with both front and rear brakes) is applied, the gross vehicle weight is used as the normal load, thus

$$F_b = \mu W\cos(\theta)$$

The coefficients S_1, S_2, and S_3 are

$$S_1 = m \tag{10a}$$

$$S_2 = W[\mu\cos(\theta) - f_o\cos(\theta) - \sin(\theta)] \tag{10b}$$

$$S_3 = \frac{\rho}{2} C_D A_f + f_1 W\cos(\theta) \tag{10c}$$

If only the **front brake** is applied, the front-wheel normal load stated in Eq. (2b) is used to determine the braking force, i.e.

$$F_b = \mu W_f = \mu \frac{W L_b\cos(\theta) - F_a h_a - mh\dot{v} - Wh\sin(\theta)}{L}$$

This leads to

$$S_1 = m\left(1 + \mu\frac{h}{L}\right) \tag{11a}$$

$$S_2 = W\left\{\mu\left[\frac{L_b}{L}\cos(\theta) - \frac{h}{L}\sin(\theta)\right] - f_o\cos(\theta) - \sin(\theta)\right\} \tag{11b}$$

$$S_3 = \frac{\rho}{2}\left(1 + \mu\frac{h_a}{L}\right) C_D A_f + f_1 W\cos(\theta) \tag{11c}$$

If only the **rear brake** is applied, the rear-wheel load stated in Eq. (2a) is used to determine the braking force, we get

$$F_b = \mu W_r = \mu \frac{W L_a \cos(\theta) + F_a h_a + m h \dot{v} + W h \sin(\theta)}{L}$$

Then S_i becomes

$$S_1 = m\left(1 - \mu \frac{h}{L}\right) \tag{12a}$$

$$S_2 = W\left\{\mu\left[\frac{L_a}{L}\cos(\theta) + \frac{h}{L}\sin(\theta)\right] - f_o \cos(\theta) - \sin(\theta)\right\} \tag{12b}$$

$$S_3 = \frac{\rho}{2}\left(1 - \mu \frac{h_a}{L}\right) C_D A_f + f_1 W \cos(\theta) \tag{12c}$$

Closed-form Solution

Eq. (8) or Eq. (9) can be solved numerically or symbolically. Integrating Eq. (9) gives

$$\int_{t_0}^{t} dt = -S_1 \int_{v_0}^{v} \frac{1}{S_2 + S_3 v^2} dv$$

The closed-form (symbolic) solution for the above equation is

$$t - t_0 = -\frac{S_1}{\sqrt{S_2 S_3}}\left[\tan^{-1}\left(\sqrt{\frac{S_3}{S_2}}\, v\right) - \tan^{-1}\left(\sqrt{\frac{S_3}{S_2}}\, v_0\right)\right] \tag{13}$$

Where v_o is the initial velocity at the initial time t_o. Assume $t_o = 0$, the equation can be simplified. And after rearrangement the velocity during braking becomes

$$v = \sqrt{\frac{S_2}{S_3}} \tan\left[\tan^{-1}\left(\sqrt{\frac{S_3}{S_2}}\, v_0\right) - \frac{\sqrt{S_2 S_3}}{S_1} t\right] \tag{14}$$

The stop time (t_s) is obtained by setting the final velocity $v = 0$, i.e.

$$\tan^{-1}\left(\sqrt{\frac{S_3}{S_2}}\, v_0\right) - \frac{\sqrt{S_2 S_3}}{S_1} t_s = 0 \tag{15}$$

Rearranging the equation gives the stop time as follows

$$t_s = \frac{S_1}{\sqrt{S_2 S_3}} \tan^{-1}\left(\sqrt{\frac{S_3}{S_2}}\, v_0\right) \tag{16}$$

The braking distance (S_d) is obtained by integrating Eq. (14)

$$S_d = \int_{t_0}^{t} v\,dt = \sqrt{\frac{S_2}{S_3}} \int_{t_0}^{t} \tan\left[\tan^{-1}\left(\sqrt{\frac{S_3}{S_2}}\, v_0\right) - \frac{\sqrt{S_2 S_3}}{S_1} t\right] dt \tag{17}$$

Denote $C_1 = \tan^{-1}\left(\sqrt{\frac{S_3}{S_2}}v_0\right)$ and $C_2 = \frac{\sqrt{S_2 S_3}}{S_1}$, the closed-form solution of braking distance becomes

$$S_d = \sqrt{\frac{S_2}{S_3}}\int_{t_0}^{t}\tan(C_1 - C_2 t)\,dt \tag{18}$$

Define $u = C_1 - C_2 t$, we get $dt = -\frac{du}{C_2}$ and

$$S_d = -\frac{1}{C_2}\sqrt{\frac{S_2}{S_3}}\int_{u_0}^{u}\tan(u)\,du$$

$$= \frac{1}{C_2}\sqrt{\frac{S_2}{S_3}}\{\log_e|\cos(C_1 - C_2 t)|\}\Big|_{t_o}^{t}$$

$$= \frac{1}{C_2}\sqrt{\frac{S_2}{S_3}}\left\{\log_e\left|\frac{\cos(C_1 - C_2 t)}{\cos(C_1 - C_2 t_o)}\right|\right\}$$

Substitute C_1 and C_2 back to the above equation, the braking distance becomes

$$S_d = \frac{S_1}{S_3}\left\{\log_e\left|\frac{\cos\left(\tan^{-1}\left(\sqrt{\frac{S_3}{S_2}}v_0\right) - \frac{\sqrt{S_2 S_3}}{S_1}t\right)}{\cos\left(\tan^{-1}\left(\sqrt{\frac{S_3}{S_2}}v_0\right)\right)}\right|\right\} \tag{19}$$

Simulation Results and Summary

The simulation is conducted in Matlab and for illustration purpose a sample program is attached in the Appendix.

Figure 2 shows the braking performance of the bike with initial velocity v_o = 60 km/hr and tire-road coefficient of friction μ = 0.8 for full-brake, front-brake, and rear-brake cases. It can be seen that the velocities decrease almost linearly due to the fact that the decelerations are nearly constants. The nonlinearity in the decelerations is due to the air drag and tire rolling resistance at high speeds; however the nonlinearity becomes weaker as the vehicle speed decreases. This implies that the effect of air drag and tire rolling resistance may be neglected at low vehicle speeds. The braking efficiency is defined as the g-normalized deceleration divided by the tire-road coefficient of friction μ (i.e. $\eta = a_g/\mu = a/g\mu$). With the inclusion of air drag and tire rolling resistance, the braking efficiency is slightly greater than unity.

Figure 3 shows the braking performance of the vehicle with various initial velocities for full-brake, front-brake, and rear-brake cases. Figure 4 shows the braking performance of the vehicle with various tire-road coefficients of friction and an initial velocity of 60 km/hr for full-brake, front-brake, and rear-brake cases. Again it can be seen that the decelerations and braking efficiencies are affected by the vehicle speed and rolling resistance.

From the values of decelerations, stop times, braking distances, and braking efficiencies, it is necessary to keep the full brakes functional in order to obtain the best braking performance.

Based on the nonlinear equation derived from force equilibrium, the closed-form solution for determining the velocity, stop time, and braking distance is obtained. With the closed-form solution, the estimation of braking performance with various vehicle parameters for full-brake, front-brake, and rear-brake cases becomes very straightforward.

References

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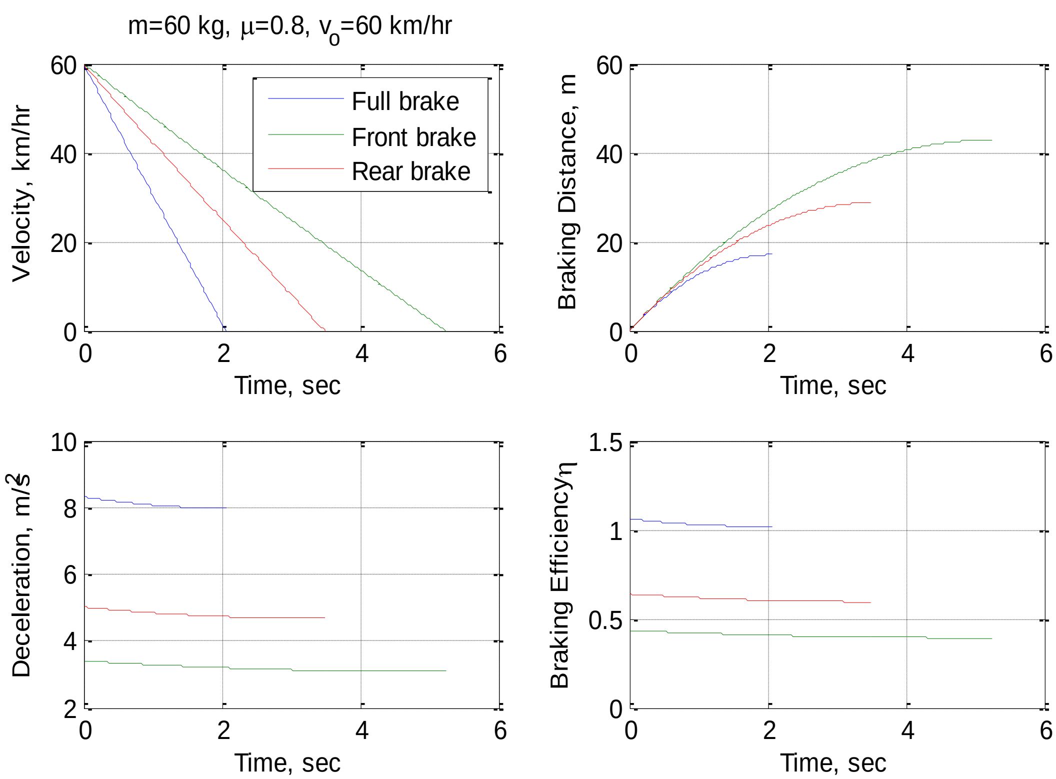

Figure 2: Braking performance of the bike with initial velocity v_o = 60 km/hr and tire-road coefficient of friction μ = 0.8 for full-brake, front-brake and rear-brake cases (note: $\eta = a/\mu g$).

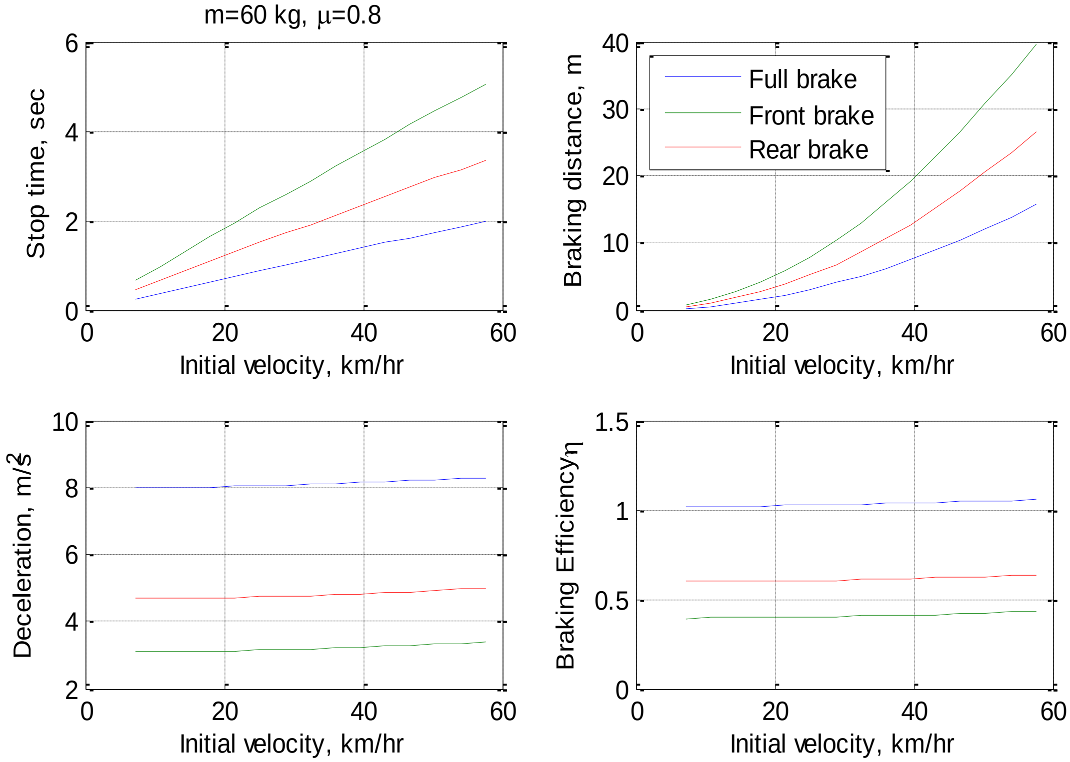

Figure 3: Braking performance of various initial velocities for full-brake, front-brake and rear-brake cases.

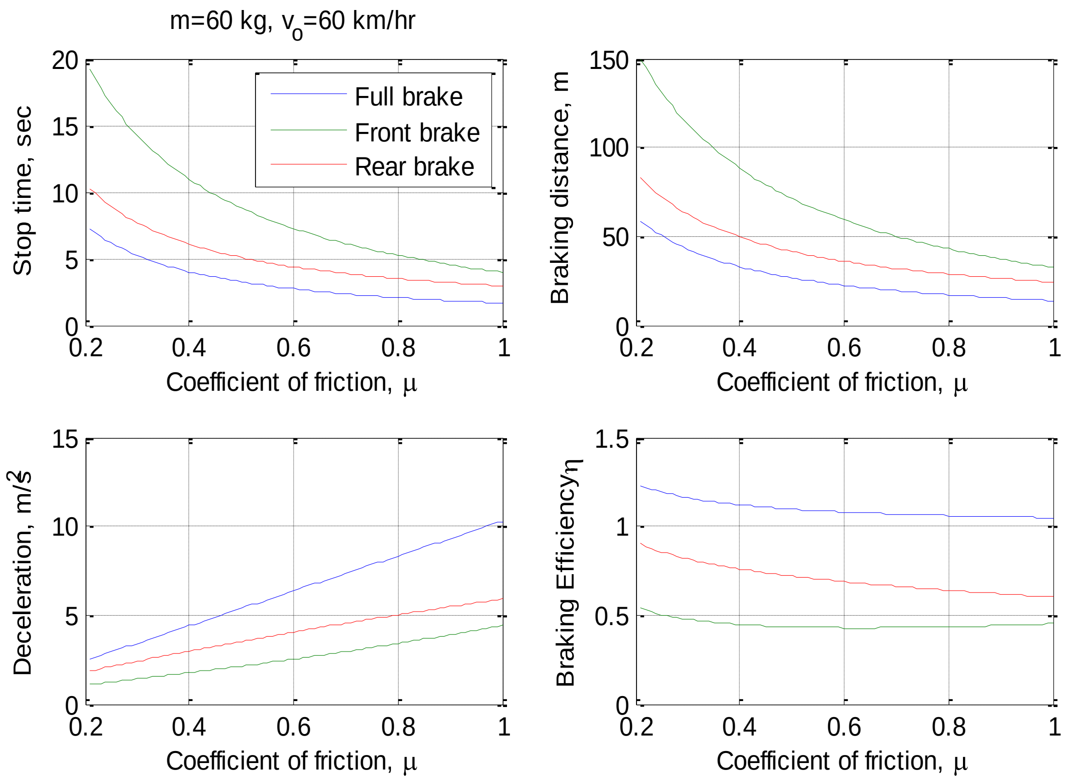

Figure 4: Braking performance of various tire-road coefficients of friction for full-brake, front-brake and rear-brake cases.

Appendix: Sample Matlab Code

```
%% closed_form_brake_bike_sample.m
% Prof. Junghsen Lieh, Wright State University, Ohio - USA
% junghsen.lieh@wright.edu, December 21, 2012
clf, clear all;
m     = 60;     den = 1.218;    Cd = 0.30;      Af = 0.40;
g     = 9.81;    fo = 0.0136;    f1 = 0.0000005184;
mu    = 0.8;      h = 0.4;       ha = 0.45;        W = m*g;
La    = 1.0;     Lb = 0.40;      L  = La + Lb;
grade = 0;   theta = atan(grade/100);
fo    = fo*cos(theta);          f1 = f1*cos(theta);  % correction for grade
% Full Brakes
s2b1 = m;                                                     s1(1) = s2b1;
s2b2 = W*(mu*cos(theta) + fo + sin(theta));                   s2(1) = s2b2;
s2b3 = (den/2)*Cd*Af + f1*W;                                  s3(1) = s2b3;
% Front Brake Only
sfb1 = m*(1 - mu*h/L);                                        s1(2) = sfb1;
sfb2 = W*(mu*(Lb/L*cos(theta)-h/L*sin(theta))+fo+sin(theta)); s2(2) = sfb2;
sfb3 = (den/2)*(1 - mu*ha/L)*Cd*Af + f1*W;                    s3(2) = sfb3;
% Rear Brake Only
srb1 = m*(1 + mu*h/L);                                        s1(3) = srb1;
srb2 = W*(mu*(La/L*cos(theta)+h/L*sin(theta))+fo+sin(theta)); s2(3) = srb2;
srb3 = (den/2)*(1 + mu*ha/L)*Cd*Af + f1*W;                    s3(3) = srb3;
% Closed-form Solution
kmhr = 60;                % Initial speed, km/hr
vo   = kmhr*1000/3600;    % Initial speed, m/s
dv   = -0.02;     npt = floor(abs((vo-0)/dv));
to   = 0;          tf = 20;                        dt = tf/npt;
for jj = 1:3
  c1     = atan(sqrt(s3(jj)/s2(jj))*vo);
  c2     = sqrt(s2(jj)*s3(jj))/s1(jj);
  ts2bT  = s1(jj)/sqrt(s2(jj)*s3(jj))*atan(sqrt(s3(jj)/s2(jj))*vo);
  ts(jj) = ts2bT;  % Stop time, sec
% Velocity v(t), m/s
  v = vo;
  ii = 0;  t = 0;
  while t < ts(jj)
    ii = ii + 1;
    t    =  to + (ii-1)*dt;
    v    = sqrt(s2(jj)/s3(jj))*tan(c1-c2*t);       % Velocity at time t, sec
    dis  = s1(jj)/s3(jj)*(log(cos(c1-c2*t)/cos(c1))); % Braking distance, m
    a    = -(s2(jj) + s3(jj)*v^2)/s1(jj);             % Deceleration, m/s^2
    if jj==1,  t2b(ii)=t;  dis2b(ii)=dis;  a2b(ii)=a;  v2b(ii)=v; end
    if jj==2,  tfb(ii)=t;  disfb(ii)=dis;  afb(ii)=a;  vfb(ii)=v; end
    if jj==3,  trb(ii)=t;  disrb(ii)=dis;  arb(ii)=a;  vrb(ii)=v; end
  end
end
rt = 3600/1000;  % Convert v from m/s to km/hr
% Plotting
subplot(2,2,1)
  plot(t2b,rt*v2b,  tfb,rt*vfb,  trb,rt*vrb), grid
    xlabel('Time, sec'),  ylabel('Velocity, km/hr')
    title(['m=',num2str(m), ' kg, \mu=', num2str(mu),...
           ', v_o=',num2str(kmhr),' km/hr'])
    ylim([0 60]), legend('Full brake', 'Front brake', 'Rear brake')
subplot(2,2,2)
  plot(t2b,dis2b,  tfb,disfb,  trb,disrb), grid
    xlabel('Time, sec'),  ylabel('Braking Distance, m')
```

```
subplot(2,2,3)
  plot(t2b,-a2b,  tfb,-afb,  trb,-arb), grid
    xlabel('Time, sec'),  ylabel('Deceleration, m/s^2')
subplot(2,2,4)
  plot(t2b,-a2b/(g*mu),  tfb,-afb/(g*mu),  trb,-arb/(g*mu)), grid
    xlabel('Time, sec'),  ylabel('Braking Efficiency, \eta')
```